International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

On the Representation of Quantum Mechanics on a Classical Sample Space

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Received July 4, 1997

Under the common viewpoint of statistical maps, the concept of observables in quantum mechanics and in classical probability theory are discussed and compared. It is shown that, by means of injective statistical maps, quantum mechanics can to a certain extent be reformulated in classical terms. Some characteristic examples are considered.

1. QUANTUM MECHANICS AND CLASSICAL PROBABILITY THEORY

Hilbert-space quantum mechanics is based on the statistical duality $\langle \mathcal{T}_s(\mathcal{H}), \mathcal{B}_s(\mathcal{H}) \rangle$ consisting of the space of the self-adjoint trace-class operators and the space of the bounded self-adjoint operators in a nontrivial, complex separable Hilbert space $\mathcal{H}$, where the duality is given by the trace functional according to $\langle V, A \rangle := \operatorname{tr} V A$. The space $\mathcal{T}_s(\mathcal{H})$ is generated by the convex set $K(\mathcal{H})$ of all density operators; $\mathcal{B}_s(\mathcal{H})$ is generated by the order-unit interval $[0, 1]$, which is also a convex set. A density operator $W \in K(\mathcal{H})$ describes a quantum state, i.e., a statistical ensemble, and an operator $A \in [0, 1]$ an *effect*, i.e., a class of statistically equivalent yes–no experiments; the number $\operatorname{tr} W A \in [0, 1]$ is interpreted to be the probability for the outcome 'yes' of the effect A in a state W.

Let M be a nonempty set, (M, Ξ) a measurable space, and $K(M, \Xi)$ the convex set of all probability measures on Ξ. An affine map $T\colon K(\mathcal{H}) \to K(M, \Xi)$ is called a *statistical map*. Such a map describes a quantum observable with value space M where TW is the probability distribution of that observable

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0020-7748/98/0100-034 9$15.00/0

in the state W. It is well known and not hard to prove that T can be represented according to

$$(TW)(B) = \text{tr } WF(B) =: P_W^F(B) \tag{1}$$

where $W \in K(\mathcal{H})$, $B \in \Xi$, and $F\colon \Xi \to [0, 1]$ is a uniquely determined normalized POV, respectively, effect-valued measure. Conversely, every such measure F defines a statistical map. We call F an *observable on* (M, Ξ) and P_W^F its *probability distribution in the state W.*

The observables on a fixed measurable space (M, Ξ) form a convex set. The more common projection-valued observables, i.e., the normalized PV measures on (M, Ξ), are extreme points of this set; however, in general there are still other extreme points. For an observable F on $(\mathbb{R}, \Xi(\mathbb{R}))$, where $\Xi(\mathbb{R})$ denotes the Borel sets of $\mathbb{R}$, the *expectation value in a state W* is given by

$$\begin{aligned} \langle F\rangle_W &:= \int \xi P_W^F(d\xi) = \int \xi \text{ tr } WF(d\xi) \\ &= \text{tr}\left(W \int \xi F(d\xi)\right) \\ &= \text{tr } WA \end{aligned} \tag{2}$$

provided that at least the first integral exists. We have even assumed that the integral $\int \text{id}_{\mathbb{R}}\, dF =: A \in \mathcal{B}_s(\mathcal{H})$ exists in the weak sense; however, F need not be a PV measure. In the sense of equation (2), every bounded, self-adjoint operator $A \in \mathcal{B}_s(\mathcal{H})$ can be interpreted as an observable.

Classical probability theory is considered here to be based on the statistical duality $\langle \mathcal{M}_{\mathbb{R}}(\Omega, \Sigma), \mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)\rangle$ consisting of the space of the bounded signed measures and the space of the bounded measurable functions on a nontrivial measurable space (Ω, Σ), where the duality is given by the integral according to $\langle \nu, f\rangle := \int f\, d\nu$ and Ω is interpreted as a classical sample space with the elements of Σ as events. In contrast to the statistical duality of quantum mechanics, $\mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)$ is in general a proper subspace of the dual Banach space $\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)'$. The space $\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)$ is generated by the convex set $K(\Omega, \Sigma)$ of all probability measures on Σ, $\mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)$ is generated by the order-unit interval $[0, \chi_\Omega]$, where $\chi_\Omega(\omega) := 1$ for all $\omega \in \Omega$. The probability for the outcome 'yes' of an *effect* $f \in [0, \chi_\Omega]$ in a classical ensemble $\mu \in K(\Omega, \Sigma)$ is $\int f\, d\mu$; the particular effects given by the characteristic functions χ_A, $A \in \Sigma$, correspond to the events.

Analogously to the quantum case, we call an affine map $T\colon K(\Omega, \Sigma) \to K(M, \Xi)$ a *statistical map.* We call such a statistical map *regular* if $T'\mathcal{F}_{\mathbb{R}}(M, \Xi) \subseteq \mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)$, where $T'\colon \mathcal{M}_{\mathbb{R}}(M, \Xi)' \to \mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)'$ is the adjoint of the unique extension of T to a (bounded) linear map from $\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)$ into

$\mathcal{M}_{\mathbb{R}}(M, \Xi)$. It is shown in Stulpe (1986) and Bugajski *et al.* (1996) that a regular statistical map T can be represented according to

$$(T\mu)(B) = \int k(\omega, B)\, \mu(d\omega) =: P_{\mu}^{k}(B) \tag{3}$$

where $\mu \in K(\Omega, \Sigma)$, $B \in \Xi$, and k: $\Omega \times \Xi \to [0, 1]$ is a Markov kernel, i.e., $k(., B)$ is a measurable function for each $B \in \Xi$, and $k(\omega,.) \in K(M, \Xi)$ for each $\omega \in \Omega$. Conversely, every such kernel defines a regular statistical map. We interpret the Markov kernels on $\Omega \times \Xi$ as the classical observables with value space M; in this context we call a kernel k a *fuzzy random variable* and P_{μ}^{k} its *probability distribution in the statistical ensemble* μ.

Equation (3) is the analog of equation (1), and the Markov kernels are the classical analogs of the POV measures. On the one hand, $B \mapsto k(., B)$ is in fact an effect-valued measure; on the other hand, $k(\omega, .)$ is the probability distribution of the observable k in the pure ensemble $\mu = \delta_{\omega}$, where δ_{ω} denotes the Dirac measure corresponding to $\omega \in \Omega$. The fact that the measuring values of k may disperse even in an ensemble of systems in the same state ω explains the name *fuzzy* random variable; note that the measuring values of the observable k lie in M, whereas the values of the map k are numbers of $[0, 1]$.

Every *standard* random variable X: $\Omega \to M$ gives rise to a Markov kernel taking only the values 0 and 1 according to

$$k^{X}(\omega, B) := \chi_{X^{-1}(B)}(\omega) = \chi_{B}(X(\omega)) = \delta_{X(\omega)}(B) = P_{\delta_{\omega}}^{X}(B) \tag{4}$$

where $P_{\delta_{\omega}}^{X}$ is the probability distribution of X in the pure ensemble δ_{ω}. If the σ-algebra Ξ contains all one-point subsets $\{x\}$ of M, different random variables yield different Markov kernels, i.e., $k^{X_1} = k^{X_2}$ implies $X_1 = X_2$. Hence, the usual random variables can be understood as particular fuzzy random variables. It is proved in Stulpe (1986) and Bugajski *et al.* (1996) that under some slight additional assumptions equation (4) defines a one–one correspondence between the random variables X: $\Omega \to M$ and the Markov kernels k_0: $\Omega \times \Xi \to [0, 1]$ taking only the values 0 and 1 and that, moreover, these kernels k_0 are just the extreme points of the convex set of all Markov kernels k: $\Omega \times \Xi \to [0, 1]$. Thus, the standard random variables can be viewed as the extreme points of the convex set of all fuzzy random variables with the same outcome space.

2. CLASSICAL REPRESENTATIONS

We call an injective statistical map T: $K(\mathcal{H}) \to K(\Omega, \Sigma)$ a *classical representation of quantum mechanics on* (Ω, Σ). A statistical map T: $K(\mathcal{H}) \to$

$K(\Omega, \Sigma)$ is injective if and only if the observable F on (Ω, Σ) that corresponds to T according to (1) separates the states, i.e., if and only if for any two $W_1, W_2 \in K(\mathcal{H})$, $P^F_{W_1} = P^F_{W_2}$ implies $W_1 = W_2$. Such observables are called *statistically complete* or *informationally complete* (e.g., Ali and Prugovečki, 1977a,b). The remarkable fact that statistically complete observables, respectively, classical representations of quantum mechanics, do exist can be concluded from the norm-separability of Hilbert space, as we are going to show.

According to the Banach–Alaoglu theorem, the closed unit ball $[-1, 1]$ of $\mathcal{B}_s(\mathcal{H}) = \mathcal{T}_s(\mathcal{H})'$ is $\mathcal{T}_s(\mathcal{H})$-weakly compact. Furthermore, because of the norm-separability of $\mathcal{T}_s(\mathcal{H})$, which is a consequence of the norm-separability of $\mathcal{H}$, the $\mathcal{T}_s(\mathcal{H})$-weak topology is metrizable *on* $[-1, 1]$. Hence, as a metrizable compact space, $[-1, 1]$ is $\mathcal{T}_s(\mathcal{H})$-weakly separable; likewise, the order-unit interval $[0, 1]$ is $\mathcal{T}_s(\mathcal{H})$-weakly separable. Now take a sequence $\tilde{A}_n \in [0, 1]$ being $\mathcal{T}_s(\mathcal{H})$-weakly dense in $[0, 1]$ and define a further sequence by

$$A_1 := 1 - \sum_{j=1}^{\infty} \frac{1}{2^j} \tilde{A}_j$$

$$A_n := \frac{1}{2^{n-1}} \tilde{A}_{n-1} \qquad \text{for} \quad n \geq 2$$

Then $A_n \in [0, 1]$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} A_n = 1$ hold, where the sum is even norm-convergent, and the linear hull of all A_n is $\mathcal{T}_s(\mathcal{H})$-weakly dense in $\mathcal{B}_s(\mathcal{H})$. Hence, according to

$$TW := (\text{tr } WA_1, \text{tr } WA_2, \ldots)$$

a classical representation T on $(\mathbb{N}, \Xi(\mathbb{N}))$ is defined, where $W \in K(\mathcal{H})$ and the probability vector TW is considered as a probability measure on the power set $\Xi(\mathbb{N})$ of the discrete sample space $\mathbb{N}$. The corresponding statistically complete observable F on $(\mathbb{N}, \Xi(\mathbb{N}))$ is given by

$$F(B) = \sum_{j \in B} A_j$$

where $B \in \Xi(\mathbb{N})$.

Now let T be an arbitrary classical representation on (Ω, Σ), respectively, its unique extension to a linear map from $\mathcal{T}_s(\mathcal{H})$ into $\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)$, and let T': $\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)' \to \mathcal{B}_s(\mathcal{H})$ be the adjoint of T. From the fact that $T' \mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)$ is a $\mathcal{T}_s(\mathcal{H})$-dense subspace of $\mathcal{B}_s(\mathcal{H})$, the validity of the following theorem follows (Singer and Stulpe, 1992).

Theorem. For every bounded, self-adjoint operator $A \in \mathcal{B}_s(\mathcal{H})$, every $\varepsilon > 0$, and any finitely many states $W_1, \ldots, W_m \in K(\mathcal{H})$ there exists a function $f \in \mathcal{F}_{\mathbb{R}}(\Omega, \Sigma)$ such that

$$\left|\text{tr } W_i A - \int f \, d\mu_i\right| < \varepsilon$$

holds, where $\mu_i := TW_i = P^F_{W_i}$ $(i = 1, \ldots, m)$.

This result signifies a far-reaching reformulation of the statistical scheme of quantum mechanics in terms of the classical sample space Ω. Namely, probabilities and expectation values which appear in reality as relative frequencies and mean values can be calculated on the basis of Hilbert space and in principle also on the basis of Ω, the latter involving real-valued standard random variables f.

Quantum dynamics can also be reformulated classically. The time development of a state $W \in K(\mathcal{H})$ is given by some Hamiltonian H according to

$$t \mapsto W_t := \tau_t W := e^{-iHt} W e^{iHt} \tag{5}$$

where $t \in \mathbb{R}$ and τ_t is a strongly continuous one-parameter group of automorphisms of the state space $\mathcal{T}_s(\mathcal{H})$. Introducing the infinitesimal generator Z of τ_t, we find that (5) satisfies

$$W_t = ZW_t \tag{6}$$

provided that the initial state W belongs to the domain $D(Z)$ of Z [for an explicit characterization of $D(Z)$ and Z, see Davies (1976); we remark that $K(\mathcal{H}) \cap D(Z)$ is dense in $K(\mathcal{H})$]. Using a classical representation T on (Ω, Σ), we can write (5) in the equivalent form

$$t \mapsto \mu_t := TW_t = T\tau_t T^{-1}\mu = \delta_t \mu \tag{7}$$

where T^{-1} is defined on the range $T\mathcal{T}_s(\mathcal{H})$ of T, $\mu := TW$, and $\delta_t = T\tau_t T^{-1}$. Note that in general $T\mathcal{T}_s(\mathcal{H})$ is not a closed subspace of $\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)$ and that the operators T^{-1} and δ_t need not be bounded; however, (7) is a solution of the equation

$$\dot{\mu}_t = L\mu_t \tag{8}$$

where the derivative is taken in the total-variation norm of $\mathcal{M}_{\mathbb{R}}(\Omega, \Sigma)$ and $L := TZT^{-1}$. Equation (8) is the reformulation of the von Neumann equation (6) in terms of the sample space Ω (for details, see Stulpe, 1996).

In the case of a finite-dimensional Hilbert space our results can be sharpened considerably. If $n := \dim \mathcal{H} < \infty$, then $\mathcal{T}_s(\mathcal{H}) = \mathcal{B}_s(\mathcal{H})$ and $\dim \mathcal{B}_s(\mathcal{H}) = n^2 =: N$. It is proved in Busch *et al.* (1993) that there exist linearly independent operators $A_1, \ldots, A_N \in \mathcal{B}_s(\mathcal{H})$ fulfilling $A_j \geq 0$ and $\Sigma_{j=1}^N A_j = 1$. Hence, according to

$$TW := (\text{tr } WA_1, \ldots, \text{tr } WA_N) \tag{9}$$

a classical representation T on $(\Omega, \Xi(\Omega))$ is defined where $W \in K(\mathcal{H})$ and the probability vector TW is considered as a probability measure on the power set $\Xi(\Omega)$ of the finite discrete sample space $\Omega := \{1, \ldots, N\}$. Although T does obviously not map $K(\mathcal{H})$ onto $K(\Omega, \Xi(\Omega))$, the linear extension of T is bijective, and so is T': $\mathcal{F}_{\mathbb{R}}(\Omega, \Xi(\Omega)) \to \mathcal{B}_s(\mathcal{H})$. Every classical representation of finite-dimensional quantum mechanics with a bijective linear extension is of the form (9), where $A_1, \ldots, A_N$ is a basis of $\mathcal{B}_s(\mathcal{H})$ as specified above. In particular, for every $A \in \mathcal{B}_s(\mathcal{H})$ there exists a uniquely determined function $a \in \mathcal{F}_{\mathbb{R}}(\Omega, \Xi(\Omega))$, namely $a := (T')^{-1} A$, such that for all $W \in K(\mathcal{H})$

$$\text{tr } WA = \sum_{j=1}^{N} p_j a_j$$

holds, where p_j are the components of the probability vector $p := TW$ and a_j are the values of the discrete standard random variable a.

A classical reformulation of finite-dimensional quantum mechanics related to ours was given by Coecke (1995). For the general case, Beltrametti and Bugajski (1995a, b) presented a classical reformulation that is based on the representation of quantum observables, respectively, POV measures by fuzzy random variables, in contrast to ours, which is based on the representation of quantum states by probability measures. Namely, if Ω is supposed to be the set of all pure states of $K(\mathcal{H})$, equipped with some suitable topology and the corresponding Borel structure $\Xi(\Omega)$, then an injective affine map from the observables F on (M, Ξ) into the fuzzy random variables k: $\Omega \times \Xi \to [0, 1]$ is defined by

$$k(P_\psi, B) := \text{tr } P_\psi F(B) = \langle\psi|F(B)\psi\rangle$$

where $P_\psi := |\psi\rangle\langle\psi|$, $\psi \in \mathcal{H}$, $\|\psi\| = 1$, and $B \in \Xi$.

3. SOME EXAMPLES

First, for spinless particles moving in one spatial direction, we consider *classical representations on phase space.* Let $u \in \mathcal{H} := L^2_{\mathbb{C}}(\mathbb{R}, dx)$ be a function of norm one and define $u_{qp}(x) := e^{ipx}u(x - q)$ for $q, p \in \mathbb{R}$. It is well known that a covariant *joint position–momentum observable* F on the phase space $\mathbb{R}^2$ is defined by the weak integral

$$F(B) := \frac{1}{2\pi} \int_B |u_{qp}\rangle\langle u_{qp}| \, dq \, dp$$

where $B \in \Xi(\mathbb{R}^2)$ is a Borel set (see, e.g., Davies, 1976); for a suitable choice of u, F is statistically complete (Ali and Prugovečki, 1977a, b). The corresponding classical representation T on $(\mathbb{R}^2, \Xi(\mathbb{R}^2))$ can be replaced by the map $\hat{T}$ assigning to each $W \in K(\mathcal{H})$ the (continuous) probability density ρ of $TW = P^F_W$, $\rho(q, p) := (1/2\pi)\langle u_{qp}|Wu_{qp}\rangle$. In particular, equation (8) can be rewritten as

$$\dot{\rho}_t = \hat{L}\rho_t$$

where the derivative is taken in the L^1-norm and the operator $\hat{L}$ is related to the classical Liouville operator $-\{H,.\}$ [e.g., Prugovečki (1984); for a rigorous discussion of the harmonic oscillator in this context, see Stulpe (1996)].

Our second example concerns *continuous classical representations* for spin-1/2 systems. Let $\mathcal{H} := \mathbb{C}^2$ and $n \cdot S$ be the operator of spin in direction $n \in \mathbb{R}^3$, $\|n\| = 1$, such that

$$n \cdot S\phi_{\pm n} = \frac{1}{2} n \cdot \sigma\phi_{\pm n} = \pm\frac{1}{2}\phi_{\pm n}$$

holds, where $\|\phi_n\| = 1$. Taking account of $|\phi_n\rangle\langle\phi_n| = \frac{1}{2}(1 + n\cdot\sigma)$, it follows that an observable F on the sphere S^2 is defined by

$$F(B) := \frac{1}{2\pi}\int_B |\phi_n\rangle\langle\phi_n|\ \kappa(dn)$$

where $B \in \Xi(S^2)$ is a Borel set and κ the rotationally invariant measure on S^2 normalized to its area; obviously, F is a statistically complete covariant *joint spin observable* (e.g., Schroeck, 1982). Again, the corresponding classical representation can be characterized by a map $\hat{T}$ assigning to each $W \in K(\mathcal{H})$ a (continuous) probability density ρ. The adjoint of $\hat{T}$, $\hat{T}': L^\infty_{\mathbb{R}}(S^2, \kappa) \to \mathcal{B}_s(\mathcal{H})$, is explicitly given by

$$\hat{T}'f = \frac{1}{2\pi}\int f(n)|\phi_n\rangle\langle\phi_n|\ \kappa(dn) \tag{10}$$

Since $\hat{T}$ is injective and $\mathcal{B}_s(\mathcal{H})$ finite-dimensional, $\hat{T}'$ is surjective. Hence, we have proved that every $A \in \mathcal{B}_s(\mathcal{H})$ can be represented according to (10).

Finally, we consider *discrete classical representations* for spin-1/2 systems. Let $n_1, n_2, n_3, n_4 \in \mathbb{R}^3$ be four unit vectors with directions determining the vertices of a regular tetrahedron and define $A_j := \frac{1}{2}|\phi_{nj}\rangle\langle\phi_{nj}| = \frac{1}{4}(1 + n_j \cdot \sigma)$. From $\Sigma_{j=1}^4 n_j = 0$ we obtain $\Sigma_{j=1}^4 A_j = 1$; hence, the positive operators A_1, A_2, A_3, A_4 constitute a discrete observable F. The probability distribution of F in the states ϕ_{n_1} and ϕ_{-n_1}, for instance, is given by the probability vectors $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, respectively; in particular, F is a statistically complete *discrete joint spin observable.* The corresponding classical representation T is, together with those obtained by orthogonal transformation of the directions n_j, distinguished under all classical representations of the form (9) with $N = 4$ since the image $TK(\mathcal{H})$ is a three-dimensional Euclidean ball inside of the tetrahedron of all probability vectors of $\mathbb{R}^4$, centered at $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and being tangent to that tetrahedron.

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